Solution to Assignment 5, MMAT5520

by YU, Rongfeng

Exercise 6.2:
1(a). **Soution:** Solving the characteristic equation

$$\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} = 0,$$

$$\lambda^2 - 2\lambda - 3 = 0,$$

$$\lambda = -1, 3.$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 3$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

respectively. Therefore the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

1(c). **Soution:** Solving the characteristic equation

$$\begin{vmatrix} \lambda-1 & 5 \\ -1 & \lambda+1 \end{vmatrix} = 0,$$

$$\lambda^2 + 4 = 0,$$

$$\lambda = \pm 2i.$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = 2i$ and $\lambda_2 = -2i$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i$$

$$\xi^{(2)} = \begin{pmatrix} 1-2i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} i$$

respectively. Therefore

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t = \begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t = \begin{pmatrix} 2\cos 2t + \sin 2t \\ \sin 2t \end{pmatrix}$$

are two linearly independent solutions and the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2\cos 2t + \sin 2t \\ \sin 2t \end{pmatrix} = \begin{pmatrix} (c_1 + 2c_2)\cos 2t + (c_2 - 2c_1)\sin 2t \\ c_1 \cos 2t + c_2 \sin 2t \end{pmatrix}.$$

1(f). **Soution:** Solving the characteristic equation

$$\begin{vmatrix} \lambda - 4 & -1 & -1 \\ -1 & \lambda - 4 & -1 \\ -1 & -1 & \lambda - 4 \end{vmatrix} = 0,$$

$$(\lambda - 3)^2(\lambda - 6) = 0,$$

$$\lambda = 3, 3, 6.$$

For the repeated root $\lambda_1 = \lambda_2 = 3$, there are two linearly independent eigenvectors

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda_3 = 6$, the associated eigenvector is

$$\xi^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

2. **Soution:** Solving the characteristic equation

$$\begin{vmatrix} \lambda - 9 & -5 \\ 6 & \lambda + 2 \end{vmatrix} = 0,$$

$$\lambda^2 - 7\lambda + 12 = 0,$$

$$\lambda = 3, 4.$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = 3$ and $\lambda_2 = 4$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

respectively. Therefore the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 5 \\ -6 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since $x_1(0) = 1, x_2(0) = 0$, we have $c_1 = -1,\ c_2 = 6$. So

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6e^{4t} - 5e^{3t} \\ -6e^{4t} + 6e^{3t} \end{pmatrix}.$$

3(b). **Soution:** Solving the characteristic equation

$$\begin{vmatrix} \lambda-1 & 1 \\ -5 & \lambda+1 \end{vmatrix} = 0,$$

$$\lambda^2 + 4 = 0,$$

$$\lambda = \pm 2i.$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = 2i$ and $\lambda_2 = -2i$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} i$$

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \end{pmatrix} i$$

respectively. Therefore

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t = \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t = \begin{pmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix}$$

are two linearly independent solutions and the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix}.$$

3(d). **Soution:** Solving the characteristic equation

$$\begin{vmatrix} \lambda-4 & 1 & 1 \\ -1 & \lambda-2 & 1 \\ -1 & 1 & \lambda-2 \end{vmatrix} = 0,$$

$$(\lambda-2)(\lambda-3)^2 = 0,$$

$$\lambda = 2, 3, 3.$$

For $\lambda_1 = 2$, the associated eigenvector is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the repeated root $\lambda_2 = \lambda_3 = 3$, there are two linearly independent eigenvectors

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Exercise 6.3:

1(a).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-1 & -2 \\ 2 & \lambda+3 \end{vmatrix} = 0,$$

$$(\lambda+1)^2 = 0,$$

$$\lambda = -1.$$

$\lambda = -1$ is a double root and the eigenspace associated with $\lambda = -1$ is of dimension 1 and is spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus

$$x^{(1)} = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution. Next, we will find a generalized eigenvector of rank 2. Take $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\eta_1 = (A+I)\eta = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (A+I)^2\eta = \mathbf{0}.$$

Thus, η is a generalized eigenvector of rank 2. Hence

$$x^{(2)} = e^{-t}(\eta + t\eta_1) = e^{-t}\begin{pmatrix} 1+2t \\ -2t \end{pmatrix}$$

is another solution to the system. Therefore the general solution is

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1+2t \\ -2t \end{pmatrix}.$$

1(c).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{vmatrix} = 0,$$

$$(\lambda-2)^2 = 0,$$

$$\lambda = 2.$$

$\lambda = 2$ is a double root and the eigenspace associated with $\lambda = 2$ is of dimension 1 and is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus

$$x^{(1)} = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a solution. Next, we will find a generalized eigenvector of rank 2. Take $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\eta_1 = (A-2I)\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (A - 2I)^2 \eta = \mathbf{0}.$$

Thus, η is a generalized eigenvector of rank 2. Hence

$$x^{(2)} = e^{2t}(\eta + t\eta_1) = e^{2t} \begin{pmatrix} 1+t \\ t \end{pmatrix}$$

is another solution to the system. Therefore the general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1+t \\ t \end{pmatrix}.$$

1(d). **Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda+3 & 0 & 4 \\ 1 & \lambda+1 & 1 \\ -1 & 0 & \lambda-1 \end{vmatrix} = 0,$$

$$(\lambda+1)^3 = 0,$$

$$\lambda = -1.$$

Thus A has an eigenvalue $\lambda = -1$ of multiplicity 3. we find that the associated eigenspace is of dimension 1 and is spanned by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. We need to find a generalized eigenvector of rank 3. Let $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then

$$\eta_1 = (A + I)\eta = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (A + I)^2\eta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_3 = (A + I)^3\eta = \mathbf{0}.$$

Therefore, η is a generalized eigenvector of rank 3. Hence

$$x^{(1)} = e^{-t}\eta_2 = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x^{(2)} = e^{-t}(\eta_1 + t\eta_2) = e^{-t} \begin{pmatrix} -2 \\ -1+t \\ 1 \end{pmatrix}$$

$$x^{(3)} = e^{-t}(\eta + t\eta_1 + \frac{t^2}{2}\eta_2) = e^{-t} \begin{pmatrix} 1-2t \\ -t+\frac{t^2}{2} \\ t \end{pmatrix}$$

form a fundamental set of solutions to the system.

Therefore the general solution is

$$\mathbf{x} = e^{-t}\left(c_1\begin{pmatrix}0\\1\\0\end{pmatrix} + c_2\begin{pmatrix}-2\\-1+t\\1\end{pmatrix} + c_3\begin{pmatrix}1-2t\\-t+\frac{t^2}{2}\\t\end{pmatrix}\right).$$

1(f). **Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix}\lambda-1 & 0 & 0\\ 2 & \lambda+2 & 3\\ -2 & -3 & \lambda-4\end{vmatrix} = 0,$$

$$(\lambda-1)^3 = 0,$$

$$\lambda = 1.$$

Thus A has an eigenvalue $\lambda = 1$ of multiplicity 3. we find that the associated eigenspace is of dimension 2 and is spanned by $\begin{pmatrix}3\\-2\\0\end{pmatrix}$ and $\begin{pmatrix}3\\0\\-2\end{pmatrix}$. Thus

$$x^{(1)} = e^t\begin{pmatrix}3\\-2\\0\end{pmatrix}, \quad x^{(2)} = e^t\begin{pmatrix}3\\0\\-2\end{pmatrix}$$

are two independent solutions. Next, We need to find a generalized eigenvector of rank 2. Let $\eta = \begin{pmatrix}1\\0\\0\end{pmatrix}$, then

$$\eta_1 = (A-I)\eta = \begin{pmatrix}0\\-2\\2\end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (A-I)^2\eta = \mathbf{0}.$$

Therefore, η is a generalized eigenvector of rank 2. Hence

$$x^{(3)} = e^t(\eta + t\eta_1) = e^t\begin{pmatrix}1\\-2t\\2t\end{pmatrix}$$

is another solution to the system.
Therefore the general solution is

$$\mathbf{x} = c_1e^t\begin{pmatrix}3\\-2\\0\end{pmatrix} + c_2e^t\begin{pmatrix}3\\0\\-2\end{pmatrix} + c_3e^t\begin{pmatrix}1\\-2t\\2t\end{pmatrix}.$$

Exercise 6.4:

1(b).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix}\lambda-5 & 4\\ -2 & \lambda+1\end{vmatrix} = 0,$$

$$\lambda^2 - 4\lambda + 3 = 0,$$

$$\lambda = 1, 3.$$

For $\lambda_1 = 1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 3$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Let $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, then we have $D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, and hence

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= -\begin{pmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{3t} - e^t & 2e^t - 2e^{3t} \\ e^{3t} - e^t & 2e^t - e^{3t} \end{pmatrix}. \end{aligned}$$

1(d).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda & -2 \\ 2 & \lambda \end{vmatrix} = 0,$$

$$\lambda^2 + 4 = 0,$$

$$\lambda = \pm 2i.$$

For $\lambda_1 = 2i$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

For $\lambda_2 = -2i$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Let $P = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, then we have $D = P^{-1}AP = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$, and hence

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{2ti} & 0 \\ 0 & e^{-2ti} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} e^{2ti} & ie^{-2ti} \\ ie^{2ti} & e^{-2ti} \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{2ti} + e^{-2ti} & -ie^{2ti} + ie^{-2ti} \\ ie^{2ti} - ie^{-2ti} & e^{2ti} + e^{-2ti} \end{pmatrix} \\ &= \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}. \end{aligned}$$

1(e).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda & -3 \\ 0 & \lambda \end{vmatrix} = 0,$$

$$\lambda^2 = 0,$$

$$\lambda = 0, 0.$$

We see that A has only one eigenvalue $\lambda = 0$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus A is not diagonalizable. So we need to find a generalized eigenvector of rank 2. Now we take $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and let

$$\eta_1 = A\eta = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\eta_2 = A^2\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 2, we may let

$$Q = [\eta_1 \quad \eta] = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$J = Q^{-1}AQ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and hence

$$\begin{aligned} e^{At} &= Qe^{Jt}Q^{-1} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{3}\begin{pmatrix} 3 & 3t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

1(g) **Soution:** $\quad e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$

2(c)**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0,$$

$$(\lambda-1)^2(\lambda+1) = 0,$$

$$\lambda = 1, 1, -1.$$

For $\lambda_1 = \lambda_2 = 1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = -1$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Let $P = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, then we have $D = P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and hence

$$\begin{aligned}
e^{At} = Pe^{Dt}P^{-1} &= \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} \\
&= -\frac{1}{2} \begin{pmatrix} 2e^{-t} & e^{t} & e^{t} \\ -e^{-t} & 0 & -e^{t} \\ e^{-t} & e^{-t} & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & 3 & 1 \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} -2e^{-t} & -2e^{-t} + 2e^{t} & -2e^{-t} + 2e^{t} \\ e^{-t} - e^{t} & e^{-t} - 3e^{t} & e^{-t} - e^{t} \\ -e^{-t} + e^{t} & -e^{-t} + e^{t} & -e^{-t} - e^{t} \end{pmatrix}.
\end{aligned}$$

Therefore the solution to the initial problem is

$$\begin{aligned}
\mathbf{x} &= e^{At}\mathbf{x}_0 \\
&= -\frac{1}{2} \begin{pmatrix} -2e^{-t} & -2e^{-t} + 2e^{t} & -2e^{-t} + 2e^{t} \\ e^{-t} - e^{t} & e^{-t} - 3e^{t} & e^{-t} - e^{t} \\ -e^{-t} + e^{t} & -e^{-t} + e^{t} & -e^{-t} - e^{t} \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} e^{t} + 2e^{-t} \\ e^{t} - e^{-t} \\ -2e^{t} + e^{-t} \end{pmatrix}.
\end{aligned}$$

Exercise 6.5:
1(c)**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda - 5 & 1 & -1 \\ -1 & \lambda - 3 & 0 \\ 3 & -2 & \lambda - 1 \end{vmatrix} = 0,$$

$$(\lambda - 3)^3 = 0,$$

$$\lambda = 3.$$

We see that A has only one eigenvalue $\lambda = 3$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Thus A is not diagonalizable. So we need to find a generalized eigenvector of rank 3. Now we take $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (A - 3I)\eta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\eta_2 = (A - 3I)^2\eta = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$\eta_3 = (A - 3I)^3\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 3, we may let

$$Q = [\eta_2 \;\; \eta_1 \;\; \eta] = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{pmatrix},$$

then

$$J = Q^{-1}AQ = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix},$$

and hence

$$\begin{aligned} e^{At} &= Qe^{Jt}Q^{-1} \\ &= \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{pmatrix} \cdot e^{3t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{pmatrix}^{-1} \\ &= e^{3t} \begin{pmatrix} 1+2t & -t & t \\ t+t^2 & 1-\frac{t^2}{2} & \frac{t^2}{2} \\ -3t+t^2 & 2t-\frac{t^2}{2} & 1-2t+\frac{t^2}{2} \end{pmatrix}. \end{aligned}$$

1(d)**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda+2 & 9 & 0 \\ -1 & \lambda-4 & 0 \\ -1 & -3 & \lambda-1 \end{vmatrix} = 0,$$

$$(\lambda-1)^3 = 0,$$

$$\lambda = 1.$$

We see that A has only one eigenvalue $\lambda = 3$, but the associated eigenspace is of dimension 2, which is spanned by $\xi^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$. Thus A is not diagonalizable. So we need to find a generalized eigenvector of rank 2. Now we take $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (A-I)\eta = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$\eta_2 = (A-I)^2\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 3, we may let

$$Q = [\xi^{(1)} \;\; \eta_1 \;\; \eta] = \begin{pmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

then

$$J = Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and hence

$$\begin{aligned} e^{At} &= Qe^{Jt}Q^{-1} \\ &= \begin{pmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \\ &= e^t \begin{pmatrix} 1-3t & -9t & 0 \\ t & 1+3t & 0 \\ t & 3t & 1 \end{pmatrix}. \end{aligned}$$

Exercise 6.6:
1(a).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda - 3 & 2 \\ -2 & \lambda + 2 \end{vmatrix} = 0,$$

$$\lambda^2 - \lambda - 2 = 0,$$

$$\lambda = -1, 2.$$

For $\lambda_1 = -1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $\lambda_2 = 2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Let $P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, then we have $D = P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.
Therefore a fundamental matrix for the system is

$$\begin{aligned} \Psi(t) = Pe^{Dt} &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}. \end{aligned}$$

1(c).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda - 2 & 5 \\ -1 & \lambda + 2 \end{vmatrix} = 0,$$

$$\lambda^2 + 1 = 0,$$

$$\lambda = \pm i.$$

For $\lambda_1 = i$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} i+2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -i$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$.

Let $P = \begin{pmatrix} i+2 & 2-i \\ 1 & 1 \end{pmatrix}$, then we have $D = P^{-1}AP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and hence

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{pmatrix} i+2 & 2-i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{ti} & 0 \\ 0 & e^{-ti} \end{pmatrix} \begin{pmatrix} i+2 & 2-i \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{2i} \begin{pmatrix} (i+2)e^{ti} & (2-i)e^{-ti} \\ e^{ti} & e^{-ti} \end{pmatrix} \begin{pmatrix} 1 & i-2 \\ -1 & i+2 \end{pmatrix} \\ &= \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}, \end{aligned}$$

which is a fundamental matrix for the system.

1(g).**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-1 & -1 & -1 \\ -2 & \lambda-1 & 1 \\ 8 & 5 & \lambda+3 \end{vmatrix} = 0,$$

$$(\lambda+1)(\lambda+2)(\lambda-2) = 0,$$

$$\lambda = -1, -2, 2.$$

For $\lambda_1 = -1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$.

For $\lambda_2 = -2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$.

For $\lambda_3 = 2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Let

$$Q = \begin{pmatrix} -3 & -4 & 0 \\ 4 & 5 & 1 \\ 2 & 7 & -1 \end{pmatrix},$$

then

$$D = Q^{-1}AQ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

Therefore a fundamental matrix for the system is

$$\begin{aligned} \Psi(t) = Qe^{Dt} &= \begin{pmatrix} -3 & -4 & 0 \\ 4 & 5 & 1 \\ 2 & 7 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-t} & -4e^{-2t} & 0 \\ 4e^{-t} & 5e^{-2t} & e^{2t} \\ 2e^{-t} & 7e^{-2t} & -e^{2t} \end{pmatrix}. \end{aligned}$$

1(j)**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-3 & -1 & -3 \\ -2 & \lambda-2 & -2 \\ 1 & 0 & \lambda-1 \end{vmatrix} = 0,$$

$$(\lambda-2)^3 = 0,$$

$$\lambda = 2.$$

We see that A has only one eigenvalue $\lambda = 2$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Thus A is not diagonalizable. So we need to find a generalized eigenvector of rank 3. Now we take $\eta = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (A-2I)\eta = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\eta_2 = (A-2I)^2\eta = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\eta_3 = (A-2I)^3\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 3, we may let

$$Q = [\eta_2 \quad \eta_1 \quad \eta] = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix},$$

then

$$J = Q^{-1}AQ = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

Therefore a fundamental matrix for the system is

$$\begin{aligned} \Psi(t) = Qe^{Jt} &= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \cdot e^{2t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} 1 & 2+t & \frac{t^2}{2}+2t+1 \\ 2 & 2+2t & t^2+2t+1 \\ -1 & -1-t & -\frac{t^2}{2}-t \end{pmatrix}. \end{aligned}$$

2(c)**Soution:** Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-3 & 0 & 0 \\ 4 & \lambda-7 & 4 \\ 2 & -2 & \lambda-1 \end{vmatrix} = 0,$$

$$(\lambda-3)^2(\lambda-5)=0,$$
$$\lambda=3,3,5.$$

For $\lambda_1=\lambda_2=3$, the associated eigenvector is $\xi^{(1)}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\xi^{(2)}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

For $\lambda_3=5$, the associated eigenvector is $\xi^{(2)}=\begin{pmatrix}0\\2\\1\end{pmatrix}$.

Let $P=\begin{pmatrix}1&1&0\\1&0&2\\0&-1&1\end{pmatrix}$, then we have $D=P^{-1}AP=\begin{pmatrix}3&0&0\\0&3&0\\0&0&5\end{pmatrix}$, and hence

$$\begin{aligned}
e^{At}=Pe^{Dt}P^{-1}&=\begin{pmatrix}1&1&0\\1&0&2\\0&-1&1\end{pmatrix}\begin{pmatrix}e^{3t}&0&0\\0&e^{3t}&0\\0&0&e^{5t}\end{pmatrix}\begin{pmatrix}2&-1&2\\-1&1&-2\\-1&1&1\end{pmatrix}\\
&=\begin{pmatrix}e^{3t}&e^{3t}&0\\e^{3t}&0&2e^{5t}\\0&-e^{3t}&e^{5t}\end{pmatrix}\begin{pmatrix}2&-1&2\\-1&1&-2\\-1&1&1\end{pmatrix}\\
&=\begin{pmatrix}e^{3t}&0&0\\2e^{3t}-2e^{5t}&-e^{3t}+2e^{5t}&2e^{3t}-2e^{5t}\\e^{3t}-e^{5t}&-e^{3t}+e^{5t}&2e^{3t}-e^{5t}\end{pmatrix}.
\end{aligned}$$

Therefore the required fundamental matrix with initial condition is

$$\begin{aligned}
\Phi(t)&=e^{At}\Phi_0\\
&=\begin{pmatrix}e^{3t}&0&0\\2e^{3t}-2e^{5t}&-e^{3t}+2e^{5t}&2e^{3t}-2e^{5t}\\e^{3t}-e^{5t}&-e^{3t}+e^{5t}&2e^{3t}-e^{5t}\end{pmatrix}\begin{pmatrix}2&0&-1\\0&-3&1\\-1&1&0\end{pmatrix}\\
&=\begin{pmatrix}2e^{3t}&0&-e^{3t}\\2e^{3t}-2e^{5t}&5e^{3t}-8e^{5t}&-3e^{3t}+4e^{5t}\\-e^{5t}&5e^{3t}-4e^{5t}&-2e^{3t}+2e^{5t}\end{pmatrix}.
\end{aligned}$$